

# Hankel transform via double Hecke algebra

Ivan Cherednik <sup>†</sup>, Yavor Markov<sup>\*</sup>

<sup>†,\*</sup> *Department of Mathematics, University of North Carolina,  
Chapel Hill, NC 27599 – 3250, USA*

March, 2000

This paper is a part of the course delivered by the first author at UNC in 2000. The focus is on the advantages of the operator approach in the theory of Bessel functions and the classical Hankel transform. We start from scratch. The Bessel functions were a must for quite a few generations of mathematicians but not anymore. We mainly discuss the *master formula* expressing the Hankel transform of the product of the Bessel function by the Gaussian.

By the operator approach, we mean the usage of the Dunkl operator and the  $\mathcal{H}''$ , *double H double prime*, the rational degeneration of the double affine Hecke algebra. This includes the transfer from the symmetric theory to the nonsymmetric one, which is the key tool of the recent development in the theory of spherical and hypergeometric functions. In the lectures, the Hankel transform was preceded by the standard Fourier transform, which is of course nonsymmetric, and the Harish-Chandra transform, which is entirely symmetric.

We followed closely the notes of the lectures not yielding to the temptation of skipping elementary calculations. We do not discuss the history and generalizations. Let us give some references. The master formula is a particular case of that from [D]. Our proof is mainly borrowed from [C1] and [C2]. The nonsymmetric Hankel transform is due to C. Dunkl (see also [O,J]). We will see that it is equivalent to the symmetric one, as well as for the master formulas (see e.g. [L], Chapter 13.4.1, formula (9)). This is a special feature of the one-dimensional setup. Generally speaking, there is an implication nonsymmetric  $\Rightarrow$  symmetric, but not otherwise.

We also study the *truncated Bessel functions*, which are necessary to treat negative half-integral  $k$ , when the eigenvalue of the Gaussian with respect to the Hankel transform is infinity. They correspond to the finite-dimensional representations of the double H double prime, which are completely described in the paper. We did not find proper references but it is unlikely that these functions never appeared before. They are very good to demonstrate the operator technique.

---

<sup>†</sup> *E-mail: chered@math.unc.edu*, Partially supported by NSF grant DMS-9877048

<sup>\*</sup> *E-mail: markov@math.unc.edu*

We thank D. Kazhdan and A. Varchenko, who stimulated the paper a great deal, and CIME for the kind invitation.

## 1 L-operator

We begin with the classical operator

$$\mathcal{L} = \left(\frac{\partial}{\partial x}\right)^2 + \frac{2k}{x} \frac{\partial}{\partial x}.$$

Upon the conjugation:

$$\mathcal{L} = |x|^{-k} \mathcal{H} |x|^k, \quad \mathcal{H} = \left(\frac{\partial}{\partial x}\right)^2 + \frac{k(1-k)}{x^2}. \quad (1)$$

Here  $k$  is a complex number. Both operators are symmetric = even.

The  $\varphi$ -function is introduced as follows:

$$\mathcal{L}\varphi_\lambda(x, k) = 4\lambda^2\varphi_\lambda(x, k), \quad \varphi_\lambda(x, k) = \varphi_\lambda(-x.k), \quad \varphi_\lambda(0, k) = 1. \quad (2)$$

We will mainly write  $\varphi_\lambda(x)$  instead of  $\varphi_\lambda(x, k)$ . Since  $\mathcal{L}$  is a DO of second order, the eigenvalue problem has a two-dimensional space of solutions. The even ones form a one-dimensional subspace and the normalization condition fixes  $\varphi_\lambda$  uniquely. Indeed, the operator  $\mathcal{L}$  preserves the space of even functions holomorphic at 0. The  $\varphi_\lambda$  can be of course constructed explicitly, without any references to the general theory of ODE.

We look for a solution in the form  $\varphi_\lambda(x, k) = f(x\lambda, k)$ . Set  $x\lambda = t$ . The resulting ODE is

$$\frac{d^2 f}{dt^2}(t) + 2k \frac{1}{t} \frac{df}{dt}(t) - 4f(t) = 0, \quad \text{a Bessel-type equation.}$$

Its even normalized solution is given by the following series

$$f(t, k) = \sum_{m=0}^{\infty} \frac{t^{2m}}{m! (k + 1/2) \cdots (k - 1/2 + m)} = \Gamma(k + \frac{1}{2}) \sum_{m=0}^{\infty} \frac{t^{2m}}{m! \Gamma(k + 1/2 + m)}. \quad (3)$$

So

$$f(t, k) = \Gamma(k + \frac{1}{2}) t^{-k+\frac{1}{2}} J_{k-\frac{1}{2}}(2it).$$

The existence and convergence is for all  $t \in \mathbf{C}$  subject to the constraint:

$$k \neq -1/2 + n, \quad n \in \mathbf{Z}_+. \quad (4)$$

The symmetry  $\varphi_\lambda(x, k) = \varphi_x(\lambda, k)$  plays a very important role in the theory. Here it is immediate. In the multi-dimensional setup, it is a theorem.

Let us discuss other (nonsymmetric) solutions of (3) and (2). Looking for  $f$  in the form  $t^\alpha(1 + ct + \dots)$  in a neighborhood of  $t = 0$ , we get that the coefficients of the expansion

$$f(t) = t^{1-2k} \sum_{m=0}^{\infty} c_m t^{2m} \text{ at } t = 0$$

can be readily calculated from (3) and are well-defined for all  $k$ .

The convergence is easy to control. Generally speaking, such  $f$  are neither regular nor even. To be precise, we get even functions  $f$  regular at 0 when  $k = -1/2 - n$  for an integer  $n \geq 0$ , i.e. when (4) does not hold. These solutions cannot be normalized as above because they vanish at 0.

Note that we do not need nonsymmetric  $f$  and the corresponding  $\varphi_\lambda(x) = f(x\lambda)$  in the paper. Only even normalized  $\varphi$  will be considered. The nonsymmetric  $\psi$ -functions discussed in the next sections are of different nature.

**Lemma 1.1** (a) Let  $\mathcal{L}^\circ$  be the adjoint operator of  $\mathcal{L}$  with respect to the  $\mathbf{C}$ -valued scalar product  $\langle f, g \rangle_0 = 2 \int_0^{+\infty} f(x)g(x)dx$ . Then  $|x|^{-2k}\mathcal{L}^\circ|x|^{2k} = \mathcal{L}$ .  
(b) Setting  $\langle f, g \rangle = 2 \int_0^{+\infty} f(x)g(x)x^{2k}dx$ , the  $\mathcal{L}$  is self-adjoint with respect to this scalar product, i.e.  $\langle \mathcal{L}(f), g \rangle = \langle f, \mathcal{L}(g) \rangle$ .

*Proof.* First, the operator multiplication by  $x$  is self-adjoint. Second,  $(\frac{\partial}{\partial x})^\circ = -\frac{\partial}{\partial x}$  via integration by parts. Finally,

$$\begin{aligned} x^{-2k}\mathcal{L}^\circ x^{2k} &= x^{-2k}((\frac{d^2}{dx^2})^\circ + (\frac{\partial}{\partial x})^\circ(\frac{2k}{x}))x^{2k} = x^{-2k}((\frac{\partial}{\partial x})^2 - \frac{\partial}{\partial x}(\frac{2k}{x}))x^{2k} \\ &= x^{-2k}(x^{2k}(\frac{\partial}{\partial x})^2 + 4kx^{2k-1}\frac{\partial}{\partial x} + 2k(2k-1)x^{2k-2} - 2kx^{2k-1}\frac{\partial}{\partial x} - 2k(2k-1)x^{2k-2}) \\ &= (\frac{\partial}{\partial x})^2 + \frac{2k}{x}\frac{\partial}{\partial x} = \mathcal{L}. \end{aligned} \tag{5}$$

Therefore,  $\langle \mathcal{L}(f), g \rangle = 2 \int_0^\infty \mathcal{L}(f)g x^{2k} dx =$

$$2 \int_{\mathbf{R}_+} f \mathcal{L}^\circ(x^{2k}g) dx = 2 \int_{\mathbf{R}_+} f x^{2k} \mathcal{L} x^{-2k}(x^{2k}g) dx = \langle f, \mathcal{L}(g) \rangle. \tag{6}$$

Actually this calculation is not necessary if (1) is used. Indeed,  $\mathcal{H}^\circ = \mathcal{H}$ .  $\square$

## 2 Hankel transform

Let us define the *symmetric Hankel transform* on the space of continuous functions  $f$  on  $\mathbf{R}$  such that  $\lim_{x \rightarrow \infty} f(x)e^{cx} = 0$  for any  $c \in \mathbf{R}$ . Provided (4),

$$(\mathbb{F}_k f)(\lambda) = \frac{2}{\Gamma(k+1/2)} \int_0^{+\infty} \varphi_\lambda(x, k) f(x) x^{2k} dx. \tag{7}$$

The growth condition makes the transform well-defined for all  $\lambda \in \mathbf{C}$ , because

$$\varphi_\lambda(x, k) \sim \text{Const}(e^{2\lambda x} + e^{-2\lambda x}) \text{ at } x = \infty.$$

The latter is standard.

We switch from  $\mathbb{F}$  on functions to the transform of the operators:  $\mathbb{F}(A)(\mathbb{F}(f) = \mathbb{F}(A(f)))$ . Remark that the Hankel transform of the function is very much different from the transform of the corresponding multiplication operator. The key point of the operator technique is the following lemma.

**Lemma 2.1** *Using the upper index to denote the variable ( $x$  or  $\lambda$ ),*

$$(a) \quad \mathbb{F}(\mathcal{L}^x) = 4\lambda^2; \quad (b) \quad \mathbb{F}(4x^2) = \mathcal{L}^\lambda; \quad (c) \quad \mathbb{F}\left(4x \frac{\partial}{\partial x}\right) = -4\lambda \frac{d}{d\lambda} - 4 - 8k.$$

*Proof.* Claim (a) is a direct consequence of Lemma 1.1 (b) with  $g(x) = \varphi_\lambda(x)$  :

$$\mathbb{F}(\mathcal{L}f) = \langle \mathcal{L}f, \varphi_\lambda \rangle = \langle f, \mathcal{L}\varphi_\lambda \rangle = 4\lambda^2 \langle f, \varphi_\lambda \rangle = 4\lambda^2 \mathbb{F}(f)$$

. Claim (b) results directly from the  $x \leftrightarrow \lambda$  symmetry of  $\phi$ , namely, from the relation  $\mathcal{L}^\lambda \varphi_\lambda(x) = 4x^2 \psi_\lambda(x)$ . Concerning (c), there are no reasons, generally speaking, to expect any simple Fourier transforms for the operators different from  $\mathcal{L}$ . However in this particular case:  $[\mathcal{L}^x, x^2] = 4x \frac{\partial}{\partial x} + 2 + 4k$ . Applying  $\mathbb{F}$  to both sides and using (a), (b),  $[4\lambda^2, \mathcal{L}^\lambda/4] = \mathbb{F}(4x \frac{\partial}{\partial x}) + 2 + 4k$ . Finally

$$\mathbb{F}\left(4x \frac{\partial}{\partial x}\right) = -4\lambda \frac{d}{d\lambda} - 2 - 4k - 2 - 4k = -4\lambda \frac{d}{d\lambda} - 4 - 8k$$

Note that

$$\left[x \frac{\partial}{\partial x}, x^2\right] = 2x^2, \quad \left[x \frac{\partial}{\partial x}, \mathcal{L}^x\right] = -2\mathcal{L}^x,$$

because operators  $x \frac{\partial}{\partial x}, \mathcal{L}$  are homogeneous of degree 2 and  $-2$ . So  $e = x^2$ ,  $f = -\mathcal{L}^x/4$ , and  $h = x \frac{\partial}{\partial x} + k + 1/2 = [e, f]$  generate a representation of the Lie algebra  $sl_2(\mathbf{C})$ .  $\square$

**Theorem 2.2 (Master Formula)** *Assuming that  $\text{Re } k > -\frac{1}{2}$ ,*

$$\begin{aligned} 2 \int_0^\infty \varphi_\lambda(x) \varphi_\mu(x) e^{-x^2} x^{2k} dx &= \Gamma(k + \frac{1}{2}) e^{\lambda^2 + \mu^2} \varphi_\lambda(\mu), \\ 2 \int_0^\infty \varphi_\lambda(x) \exp(-\frac{\mathcal{L}}{4})(f(x)) e^{-x^2} x^{2k} dx &= \Gamma(k + \frac{1}{2}) e^{\lambda^2} f(\lambda), \end{aligned} \tag{8}$$

*provided the existence of  $\exp(-\frac{\mathcal{L}}{4})(f(x))$  and the integral in the second formula.*

*Proof.* The left-hand side of the first formula equals  $\Gamma(k + 1/2)\mathbb{F}(e^{-x^2}\varphi_\mu(x))$ . We set

$$\varphi_\mu^-(x) = e^{-x^2}\varphi_\mu(x), \quad \varphi_\mu^+(x) = e^{x^2}\varphi_\mu(x).$$

They are eigenfunctions of the operators

$$\mathcal{L}_- = e^{-x^2} \circ \mathcal{L} \circ e^{x^2}, \quad \mathcal{L}_+ = e^{x^2} \circ \mathcal{L} \circ e^{-x^2}.$$

To be more exact,  $\varphi_\mu^\pm$  is a unique eigenfunction of  $\mathcal{L}^\pm$  with eigenvalue  $2\mu$ , normalized by  $\varphi_\mu^\pm(0) = 1$ .

Express  $\mathcal{L}_-$  in terms of the operators from the previous lemma.

$$\begin{aligned} e^{-x^2}\left(\frac{\partial}{\partial x}\right)^2 e^{x^2} &= e^{-x^2}\left(e^{x^2}\left(\frac{\partial}{\partial x}\right)^2 + 2(2x)e^{x^2}\frac{\partial}{\partial x} + (2 + 4x^2)e^{x^2}\right) = \left(\frac{\partial}{\partial x}\right)^2 + 4x\frac{\partial}{\partial x} + 2 + 4x^2, \\ e^{-x^2}\frac{2k}{x}\frac{\partial}{\partial x}e^{x^2} &= e^{-x^2}\left(e^{x^2}\frac{2k}{x}\frac{\partial}{\partial x} + 2xe^{x^2}\frac{2k}{x}\right) = \frac{2k}{x}\frac{\partial}{\partial x} + 4k, \\ \mathcal{L}_- &= e^{-x^2}\left(\left(\frac{\partial}{\partial x}\right)^2 + \frac{2k}{x}\frac{\partial}{\partial x}\right)e^{x^2} = \mathcal{L} + 4x\frac{\partial}{\partial x} + 2 + 4k + 4x^2. \end{aligned} \quad (9)$$

Analogously,  $\mathcal{L}_+ = \mathcal{L} - 4x\frac{\partial}{\partial x} - 2 - 4k + 4x^2$ . Now we may use Lemma 2.1:

$$\begin{aligned} \mathbb{F}(\mathcal{L}_-^x) &= \mathbb{F}(\mathcal{L}^x) + \mathbb{F}(4x^2) + \mathbb{F}(4x\frac{\partial}{\partial x}) + \mathbb{F}(2 + 4k) = \\ &= 4\lambda^2 + \mathcal{L}^\lambda - 4\lambda\frac{d}{d\lambda} - 4 - 8k + 2 + 4k = \mathcal{L}_+^\lambda. \end{aligned} \quad (10)$$

Thus

$$L_+^\lambda(\mathbb{F}\varphi_\mu^-) = \mathbb{F}(\mathcal{L}_-^x)(\mathbb{F}\varphi_\mu^-) = \mathbb{F}(\mathcal{L}_+^x\varphi_\mu^-) = 2\mu\mathbb{F}(\varphi_\mu^-),$$

i.e.  $\mathbb{F}\varphi_\mu^-$  is an eigenfunction of  $\mathcal{L}_+$  with the eigenvalue  $2\mu$ . Using the uniqueness, we conclude that  $\mathbb{F}(\varphi_\mu^-)(\lambda) = C(\mu)e^{\mu^2}\varphi_\mu^+(\lambda)$  for a constant  $C(\mu)$ . However the left-hand side of the master formula is  $\lambda \leftrightarrow \mu$  symmetric as well as  $e^{\mu^2}\varphi_\mu^+(\lambda) = e^{\lambda^2+\mu^2}\varphi_\mu(\lambda)$ . So  $C(\mu) = C(\lambda) = C$ . Setting  $\lambda = 0 = \mu$ , we get the desired.

The second formula follows from the first for  $f(x) = \varphi_\mu(x, k)$ . Move  $\exp(\mu^2)$  to the left to see this. It is linear in terms of  $f(x)$  and holds for finite linear combinations of  $\varphi$  and infinite ones provided the convergence. So it is valid for any reasonable  $f$ . We skip the detail.  $\square$

### 3 Dunkl operator

The above proof is straightforward. One needs the self-duality of the Hankel transform and the commutator representation for  $x\partial/\partial x$ . The self-duality holds in the general multi-dimensional theory. The second property is more special. Also our proof does not clarify why the master formula is so simple. There is a “one-line” proof of this important formula, which can be readily generalized. It involves the *Dunkl operator*:

$$\mathcal{D} = \frac{\partial}{\partial x} - \frac{k}{x}(s - 1), \quad \text{where } s \text{ is the reflection } s(f(x)) = f(-x). \quad (11)$$

The operator  $\mathcal{D}$  is not local anymore, because  $s$  is a global operator apart from a neighborhood of  $x = 0$ . We are going to find its eigenfunctions. Generally speaking, this may create problems since we cannot use the uniqueness theorems from the theory of ODE. However everything is surprisingly smooth.

**Lemma 3.1** *Considering  $x$  as the multiplication operator,*

$$s \circ x = -x \circ s, \quad s \circ \frac{\partial}{\partial x} = -\frac{\partial}{\partial x} \circ s, \quad (12)$$

- (a)  $\mathcal{D}^2 = \mathcal{L}$  upon the restriction to even functions,
- (b)  $s \circ \mathcal{D} \circ s = -\mathcal{D}$  and  $\mathcal{D}^2$  fixes the space of even functions.

*Proof.* Indeed,  $(s \circ x)(f(x)) = s(xf(x)) = -xf(-x) = (-x \circ s)(f(x))$ . The  $\frac{\partial}{\partial x}$  is analogous. Then

$$\begin{aligned} \mathcal{D}^2 &= \left(\frac{\partial}{\partial x}\right)^2 - \frac{k}{x}(s-1)\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{k}{x}(s-1) + \frac{k}{x}(s-1)\frac{k}{x}(s-1) \\ &= \left(\frac{\partial}{\partial x}\right)^2 + \frac{k}{x}\frac{\partial}{\partial x}(s+1) - \frac{\partial}{\partial x}\frac{k}{x}(s-1) + \frac{k}{x}(s-1)\frac{k}{x}(s-1). \end{aligned} \quad (13)$$

It is simple to calculate the final formula but unnecessary. Applying (13) to symmetric (i.e. even) functions  $f(x)$ , the two last terms will vanish, because  $(s-1)(f(x)) = f(-x) - f(x) = 0$ . So  $(s+1)(f(x)) = f(-x) + f(x) = 2f(x)$ , and  $\mathcal{D}^2|_{\text{even}} = \left(\frac{\partial}{\partial x}\right)^2 + 2\frac{k}{x}\frac{\partial}{\partial x} = \mathcal{L}$ .

Claim (b) is obvious. Indeed,  $s^2 = 1$ ,  $s\frac{\partial}{\partial x}s = -\frac{\partial}{\partial x}s^2 = -dx$ , and  $s(\frac{k}{x}(s-1))s = -\frac{k}{x}s(s^2 - s) = -\frac{k}{x}(s^3 - s^2) = -\frac{k}{x}(s-1)$ . Thus  $s \circ \mathcal{D} \circ s = -\mathcal{D}$ .

By the way, this implies that  $s \circ \mathcal{D}^2 \circ s = \mathcal{D}^2$ , i.e.  $\mathcal{D}^2$  commutes with  $s$ . So we do not need an explicit formula for  $\mathcal{D}^2|_{\text{even}}$  to see that it preserves even functions.  $\square$

Let us consider the standard scalar product  $\langle f, g \rangle_0 = \int_{-\infty}^{+\infty} f(x)g(x)dx$ . Here the functions are continuous  $\mathbf{C}$ -valued continues on the real line  $\mathbf{R}$ . One may add the complex conjugation to  $g$  but we will not do this. The scalar product is non-degenerate, so adjoint operators are well-defined. We continue to use the notation  $H^\circ$  for the pairing  $\langle f, g \rangle_0$ . Let us calculate the adjoint of  $\mathcal{D}$  with respect to  $|x|^{2k}$ .

**Proposition 3.2** *Setting  $\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x)g(x)|x|^{2k}dx$ , the Dunkl operator  $\mathcal{D}$  is anti self-adjoint with respect to this scalar product, i.e.  $\langle \mathcal{D}(f), g \rangle = -\langle f, \mathcal{D}(g) \rangle$ . Equivalently,  $|x|^{-2k} \mathcal{D}^\circ |x|^{2k} = -\mathcal{D}$ .*

*Proof.* Recall that  $x^\circ = x$  and  $(\frac{\partial}{\partial x})^\circ = -\frac{\partial}{\partial x}$ , where  $x$  is considered as the multiplication operator. Then  $s^\circ = s$ :

$$\langle s(f), g \rangle_0 = \int_{-\infty}^{+\infty} f(-x)g(x)dx = \int_{+\infty}^{-\infty} f(t)g(-t)(-dt) = \langle f, s(g) \rangle_0$$

for  $t = -x$ . Hence,

$$\begin{aligned}
|x|^{-2k} \mathcal{D}^\circ |x|^{2k} &= |x|^{-2k} \left( \frac{\partial}{\partial x} - \frac{k}{x} (s-1) \right)^\circ |x|^{2k} = |x|^{-2k} \left( \left( \frac{\partial}{\partial x} \right)^\circ - (s-1)^\circ \left( \frac{k}{x} \right)^\circ \right) |x|^{2k} \\
&= |x|^{-2k} \left( -\frac{\partial}{\partial x} - (s-1) \frac{k}{x} \right) |x|^{2k} = |x|^{-2k} \left( -\frac{\partial}{\partial x} + \frac{k}{x} (1+s) \right) |x|^{2k} \\
&= |x|^{-2k} |x|^{2k} \left( -\frac{\partial}{\partial x} \right) + |x|^{-2k} \left( -\frac{2k}{x} |x|^{2k} \right) + |x|^{-2k} |x|^{2k} \frac{k}{x} (1+s) \\
&= -\frac{\partial}{\partial x} + \frac{k}{x} (s-1) = -\mathcal{D}.
\end{aligned} \tag{14}$$

Finally,

$$\begin{aligned}
\langle \mathcal{D}(f), g \rangle &= \int_{-\infty}^{+\infty} \mathcal{D}(f(x)) g(x) |x|^{2k} dx = \int_{-\infty}^{+\infty} f(x) \mathcal{D}^\circ (|x|^{2k} g(x)) dx \\
&= \int_{-\infty}^{+\infty} f(x) |x|^{2k} (|x|^{-2k} \mathcal{D}^\circ |x|^{2k}) (g(x)) dx \\
&= \int_{-\infty}^{+\infty} f(x) (-\mathcal{D}(g(x))) |x|^{2k} dx = -\langle f, \mathcal{D}(g) \rangle. \quad \square
\end{aligned} \tag{15}$$

The proposition readily gives that  $|x|^{-2k} \mathcal{L}^\circ |x|^{2k} = \mathcal{L}$  on even functions  $f$ . Indeed,

$$\langle \mathcal{L}(f), g \rangle = \langle \mathcal{D}^2(f), g \rangle = \langle f, \mathcal{D}^2(g) \rangle = \langle f, \mathcal{L}(g) \rangle,$$

provided that  $g$  is even too. Recall that it was not difficult to check this relation directly. In the multi-dimensional theory, this calculation is more involved and the usage of the (generalized) Dunkl operators makes perfect sense.

## 4 Nonsymmetric eigenfunctions

Our next step will be a study of the eigenfunctions of the Dunkl operator:

$$\mathcal{D}\psi_\lambda(x, k) = 2\lambda\psi_\lambda(x, k), \quad \psi_\lambda(0, k) = 1. \tag{16}$$

We will use the shortcut notation  $f^\iota(x) = s(f(x)) = f(-x)$ .

**Lemma 4.1** *There exists a unique solution of the eigenfunction problem (16) for  $\lambda \neq 0$ . It is represented in the form  $\psi_\lambda(x) = g(\lambda x)$ . In the case  $\lambda = 0$ , the solution is given by the formula  $\psi_0 = 1 + Cx|x|^{-2k-1}$ , where  $C \in \mathbf{C}$  is an arbitrary constant.*

*Proof.* Assuming that  $\psi_\lambda$  is a solution of (16), let

$$\psi_\lambda^0 = \frac{1}{2}(\psi_\lambda + \psi_\lambda^\iota), \quad \psi_\lambda^1 = \frac{1}{2}(\psi_\lambda - \psi_\lambda^\iota),$$

be its even and odd parts. By Lemma 3.1 (b),  $\mathcal{D}s(\psi_\lambda(x)) = -s\mathcal{D}(\psi_\lambda(x)) = -2\lambda s(\varphi_\lambda(x))$ . Hence, (16) is equivalent to

$$\begin{aligned}\mathcal{D}\psi_\lambda^0 &= 2\lambda\psi_\lambda^1; & \psi_\lambda^0(0) &= 1 \\ \mathcal{D}\psi_\lambda^1 &= 2\lambda\psi_\lambda^0 & \psi_\lambda^1(0) &= 0.\end{aligned}\tag{17}$$

Furthermore,  $\mathcal{D}^2\psi_\lambda^0 = 4\lambda^2\psi_\lambda^0$ . Since  $\psi_\lambda^0$  is even,  $\mathcal{L}\psi_\lambda^0 = 4\lambda^2\psi_\lambda^0$  due to Lemma 3.1. Therefore  $\psi_\lambda^0$  has to coincide with  $\varphi_\lambda$  from the first section. This is true for all  $\lambda$ . If  $\lambda \neq 0$ ,

$$\psi_\lambda^1 = \frac{1}{2\lambda}\mathcal{D}\psi_\lambda^0 = \frac{1}{2\lambda}\left(\frac{d\psi_\lambda^0}{dx} - \frac{k}{x}(s-1)\psi_\lambda^0\right) = \frac{1}{2\lambda}\frac{d\varphi_\lambda}{dx}.\tag{18}$$

The last equality holds because  $\psi_\lambda^0 = \varphi_\lambda$  is even. Finally,

$$\psi_\lambda(x) = \varphi_\lambda(x) + \frac{1}{2\lambda}\varphi'_\lambda(x) = g(\lambda x) \text{ for } g = f + \frac{1}{2}f',\tag{19}$$

where  $\varphi_\lambda(x) = f(\lambda x)$ ,  $f$  is from (3), and  $f'$  is the derivative. It is for  $\lambda \neq 0$ .

Let us consider the case  $\lambda = 0$ . We have  $\mathcal{D}\psi_\lambda^0 = 0$ ,  $\psi_\lambda^0(0) = 1$  and  $\mathcal{D}\psi_\lambda^1 = 0$ ,  $\psi_\lambda^1(0) = 0$ . Since  $\psi_\lambda^0$  is even,  $\mathcal{D}\psi_\lambda^0 = \frac{d\psi_\lambda^0}{dx} = 0$ . Thus  $\psi_\lambda^0(x) = 1$ . The  $\psi_\lambda^1$  is odd. So

$$\mathcal{D}\psi_\lambda^1(x) = \frac{d\psi_\lambda^1(x)}{dx} - \frac{k}{x}(s-1)\psi_\lambda^1(x) = \frac{d\psi_\lambda^1(x)}{dx} + \frac{k}{x}(\psi_\lambda^1(x) - \psi_\lambda^1(-x)) = \frac{d\psi_\lambda^1(x)}{dx} + \frac{2k}{x}\psi_\lambda^1(x).$$

Solving the resulting ODE,  $\psi_\lambda^1(x) = Cx|x|^{-2k-1}$ . □

In this proof, we used that  $\mathcal{D}f(x) = f'(x)$  on even functions and

$$\mathcal{D}f(x) = f'(x) + \frac{k}{x}(f(x) - f(-x)) = f'(x) + \frac{2k}{x}f(x) = \left(\frac{\partial}{\partial x} + \frac{2k}{x}\right)f(x)$$

on odd functions. By the way, it makes obvious the coincidence of  $\mathcal{L}$  with  $\mathcal{D}^2$  on even  $f$ . Indeed,  $\mathcal{D}^2f(x) = \left(\frac{\partial}{\partial x} + \frac{2k}{x}\right)\left(\frac{\partial}{\partial x}f(x)\right)$ . For odd  $f$ , it is the other way round:  $\mathcal{D}^2f(x) = \mathcal{D}(\mathcal{D}f(x)) = \frac{\partial}{\partial x}(\mathcal{D}f(x)) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + \frac{2k}{x}\right)(f(x))$ . In particular,

$$\mathcal{D}^2\psi_\lambda^1(x) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + \frac{2k}{x}\right)\psi_\lambda^1(x) = \left(\left(\frac{\partial}{\partial x}\right)^2 + \frac{\partial}{\partial x}\frac{2k}{x}\right)\psi_\lambda^1(x).$$

Hence  $\psi_\lambda^1(x)$  is also a solution of a second order differential equation. This equation is different from that for  $\varphi$ , but not too different. Comparing them we come to the important definition of the *shift operator*. We show the dependence of  $\mathcal{L}$  on  $k$  and set  $\widetilde{\mathcal{L}}_k = \left(\frac{\partial}{\partial x}\right)^2 + \frac{\partial}{\partial x}\frac{2k}{x}$ .

**Lemma 4.2** (a)  $x^{-1} \circ \widetilde{\mathcal{L}}_k \circ x = \mathcal{L}_{k+1}$ .

(b)  $\widetilde{\mathcal{L}}_k\psi_\lambda^1 = 4\lambda^2\psi_\lambda^1$ .

(c)  $(x^{-1} \circ \widetilde{\mathcal{L}}_k \circ x)(x^{-1}\psi_\lambda^1) = 4\lambda^2(x^{-1}\psi_\lambda^1)$ .



*Proof.* The first claim:

$$\begin{aligned}
x^{-1} \circ \left(\frac{\partial}{\partial x}\right)^2 \circ x &= x^{-1} \left( x \left(\frac{\partial}{\partial x}\right)^2 + 2 \frac{\partial}{\partial x} \right) = \left(\frac{\partial}{\partial x}\right)^2 + \frac{2}{x} \frac{\partial}{\partial x}, \\
x^{-1} \circ \frac{\partial}{\partial x} \frac{2k}{x} \circ x &= x^{-1} \circ \frac{\partial}{\partial x} \circ 2k = \frac{2k}{x} \frac{\partial}{\partial x}, \\
x^{-1} \circ \widetilde{\mathcal{L}}_k \circ x &= \left(\frac{\partial}{\partial x}\right)^2 + \frac{2}{x} \frac{\partial}{\partial x} + \frac{2k}{x} \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x}\right)^2 + \frac{2(k+1)}{x} \frac{\partial}{\partial x} = \mathcal{L}_{k+1}. \tag{20}
\end{aligned}$$

Then  $\widetilde{\mathcal{L}}_k \psi_\lambda^1 = \mathcal{D}^2 \psi_\lambda^1 = 4\lambda^2 \psi_\lambda^1$  due to (17). Claim (c) is a combination of (a) and (b).  $\square$

**Proposition 4.3 (Shift Formula)**

$$\frac{1}{x} \psi_\lambda^1(x, k) = \frac{2\lambda}{1+2k} \psi_\lambda^0(x, k+1), \quad i.e. \quad \frac{1}{x} \frac{d\varphi_\lambda}{dx}(x, k) = \frac{4\lambda^2}{1+2k} \varphi_\lambda(x, k+1). \tag{21}$$

*Proof.* Lemma 4.2 (c) implies that  $x^{-1} \psi_\lambda^1(x, k) = C(\lambda, k) \varphi_\lambda(x, k+1)$ , because  $\varphi_\lambda(x, k+1)$  is a unique even normalized solution of (2) for  $k+1$ . Thanks to (18)  $\psi_\lambda^1(x, k) = (2\lambda)^{-1} \frac{d\varphi_\lambda}{dx}(x, k)$ . Thus  $x^{-1} \varphi'_\lambda(x, k) = C(\lambda, k) \varphi_\lambda(x, k+1)$ . The constant  $C$  readily results from the expansion (3) of  $\varphi_\lambda(x, k)$ . Explicitly:

$$\begin{aligned}
0 &= (\mathcal{L}_k \varphi_\lambda - 4\lambda^2 \varphi_\lambda)(0) \Rightarrow \\
0 &= (2k+1) \left( x^{-1} \frac{d\varphi_\lambda}{dx} \right)(0, k) - 4\lambda^2. \tag{22}
\end{aligned}$$

The shift formula can be of course checked directly without  $\psi_\lambda^1$ , a good exercise.  $\square$

## 5 Master formula

Let us define the *nonsymmetric Hankel transform*. We consider complex-valued  $C^\infty$ - functions  $f$  on  $\mathbf{R}$  such that  $\lim_{x \rightarrow \infty} f(x) e^{cx} = 0$  for any  $c \in \mathbf{R}$  and set

$$(\mathcal{F}f)(\lambda) = \frac{1}{\Gamma(k+1/2)} \int_{-\infty}^{+\infty} \psi_\lambda(x, k) f(x) |x|^{2k} dx \tag{23}$$

We assume that  $\operatorname{Re} k > -\frac{1}{2}$  and always take  $\psi_0(x, k) = 1$ . Recall that the case  $\lambda = 0$  is exceptional (Lemma 4.1): the dimension of the space of eigenfunctions is 2.

Let us compute the transforms of our main operators. Compare it with Lemma 2.1: it is much more comfortable to deal with the operators of the first order. The upper index denotes the variable.

**Lemma 5.1**

$$(a) \quad \mathcal{F}(\mathcal{D}^x) = -2\lambda; \quad (b) \quad \mathcal{F}(2x) = \mathcal{D}^\lambda; \quad (c) \quad \mathcal{F}(s^x) = s^\lambda.$$

*Proof.* The first formula is an immediate consequence of Proposition 3.2 (a) with  $g(x) = \psi_\lambda(x)$  :

$$\mathcal{F}(\mathcal{D}f) = \langle \mathcal{D}f, \psi_\lambda \rangle = -\langle f, \mathcal{D}\psi_\lambda \rangle = -2\lambda \langle f, \psi_\lambda \rangle = -2\lambda \mathcal{F}(f).$$

Claim (b) follows from the  $x \leftrightarrow \lambda$  symmetry. As to (c), use that  $\psi_\lambda(-x) = \psi_{-\lambda}(x)$ .  $\square$

**Theorem 5.2 (Nonsymmetric Master Formula)**

$$\begin{aligned} \int_{-\infty}^{\infty} \psi_\lambda(x) \psi_\mu(x) e^{-x^2} |x|^{2k} dx &= \Gamma(k + \frac{1}{2}) e^{\lambda^2 + \mu^2} \psi_\lambda(\mu), \\ \int_{-\infty}^{\infty} \psi_\lambda(x) \exp(-\frac{\mathcal{D}^2}{4})(f(x)) e^{-x^2} |x|^{2k} dx &= \Gamma(k + \frac{1}{2}) e^{\lambda^2} f(\lambda). \end{aligned} \quad (24)$$

*Proof.* In the first formula, the left-hand side equals  $\Gamma(k + 1/2) \mathcal{F}(e^{-x^2} \psi_\mu(x))$ . We set

$$\begin{aligned} \psi_\mu^-(x) &= e^{-x^2} \psi_\mu(x), \quad \psi_\mu^+(x) = e^{x^2} \psi_\mu(x), \\ \mathcal{D}_- &= e^{-x^2} \circ \mathcal{D} \circ e^{x^2}, \quad \mathcal{D}_+ = e^{x^2} \circ \mathcal{D} \circ e^{-x^2}. \end{aligned}$$

The function  $\psi_\mu^\pm$  is an eigenfunction of  $\mathcal{D}^\pm$  with eigenvalue  $2\mu$ . The normalization fixes it uniquely with the standard reservation about  $\mu = 0$ .

One gets:

$$\mathcal{D}_- = e^{-x^2} \left( \frac{\partial}{\partial x} - \frac{k}{x}(s-1) \right) e^{x^2} = \frac{\partial}{\partial x} + 2x - \frac{k}{x}(s-1) = \mathcal{D} + 2x.$$

Correspondingly,  $\mathcal{D}_+ = \mathcal{D} - 2x$ . Using Lemma 5.1 ,

$$\mathcal{F}(\mathcal{D}_-^x) = \mathcal{F}(\mathcal{D}^x) + \mathcal{F}(2x) = -2\lambda + \mathcal{D}^\lambda = \mathcal{D}_+^\lambda.$$

Therefore

$$D_+^\lambda(\mathcal{F}\psi_\mu^-) = \mathcal{F}(\mathcal{D}_-^x)(\mathcal{F}\psi_\mu^-) = \mathcal{F}(\mathcal{D}_-^x \psi_\mu^-) = 2\mu \mathcal{F}(\psi_\mu^-),$$

i.e.  $\mathcal{F}(\psi_\mu^-)$  is an eigenfunction of  $\mathcal{D}_+$  with the eigenvalue  $2\mu$ , and  $\mathcal{F}(\psi_\mu^-)(\lambda) = C(\mu) e^{\mu^2} \psi_\mu^+(\lambda)$ . Using the  $\lambda \leftrightarrow \mu$ -symmetry,  $C(\mu) = C(\lambda) = C$  and

$$C = \int_{\mathbf{R}} e^{-x^2} |x|^{2k} dx = \Gamma(k + \frac{1}{2}).$$

Cf. the proof of the symmetric master formula.

The second formula readily follows from the first provided the existence of the function  $\exp(-\frac{\mathcal{D}^2}{4})(f(x))$  and the corresponding integral. The latter function has to go to zero at  $x = \infty$  faster than  $e^{cx}$  for any  $c \in \mathbf{R}$ .  $\square$

The symmetric master theorem is of course a particular case of (24). Indeed, we may replace  $\psi_\lambda(x)$  by  $2\varphi_\lambda(x) = \psi_\lambda(x) + \psi_\lambda(-x) = \psi_\lambda(x) + \psi_{-\lambda}(x)$  on the left-hand side. Then

$$\psi_\lambda(\mu) \mapsto \psi_\lambda(\mu) + \psi_{-\lambda}(\mu) = \psi_\lambda(\mu) + \psi_\lambda(-\mu) = 2\varphi_\lambda(\mu)$$

on the right-hand side. We use that the factor  $e^{\lambda^2+\mu^2}$  is even. Now we either repeat the same transfer for  $\mu$  or simply symmetrize the integrand.

It is more surprising that the nonsymmetric theorem can be deduced from the symmetric one. It is a special feature of the one-dimensional case. Generally speaking, there is no reason to expect such an implication. This may clarify why the nonsymmetric Hankel transform and  $\psi$  were of little importance in the classical theory of Bessel functions. They could be considered as a minor technical improvement of the symmetric theory. Now we have the opposite point of view.

Let us deduce Theorem 5.2 from Theorem 2.2. We may assume that  $\lambda, \mu \neq 0$ . Discarding the odd summands in the integrand,

$$\begin{aligned} \int_{\mathbf{R}} \psi_{\lambda} \psi_{\mu} e^{-x^2} |x|^{2k} dx &= \int_{\mathbf{R}} (\psi_{\lambda}^0 + \psi_{\lambda}^1)(\psi_{\mu}^0 + \psi_{\mu}^1) e^{-x^2} |x|^{2k} dx \\ &= \int_{\mathbf{R}} (\psi_{\lambda}^0 \psi_{\mu}^0 + \psi_{\lambda}^1 \psi_{\mu}^1) e^{-x^2} |x|^{2k} dx = \int_{\mathbf{R}} (\varphi_{\lambda} \varphi_{\mu} + \psi_{\lambda}^1 \psi_{\mu}^1) e^{-x^2} |x|^{2k} dx. \end{aligned} \quad (25)$$

The integral of  $\varphi_{\lambda} \varphi_{\mu}$  is nothing else but (8). Let us use the shift formula to manage  $\psi_{\lambda}^1 \psi_{\mu}^1$ . See Proposition 4.3.

We get  $(\psi_{\lambda}^1 \psi_{\mu}^1)(x, k) = \frac{4\lambda\mu x^2}{(1+2k)^2} (\varphi_{\lambda} \varphi_{\mu})(x, k+1)$  and

$$\begin{aligned} \int_{\mathbf{R}} (\psi_{\lambda}^1 \psi_{\mu}^1)(x, k) e^{-x^2} |x|^{2k} dx &= \frac{4\lambda\mu}{(1+2k)^2} \int_{\mathbf{R}} (\varphi_{\lambda} \varphi_{\mu})(x, k+1) e^{-x^2} |x|^{2(k+1)} dx \\ &= \frac{4\lambda\mu}{(1+2k)^2} \varphi_{\lambda}(\mu, k+1) e^{\lambda^2+\mu^2} \Gamma(k + \frac{3}{2}) = \frac{2\lambda\mu}{(1+2k)} \varphi_{\lambda}(\mu, k+1) e^{\lambda^2+\mu^2} \Gamma(k + \frac{1}{2}) \\ &= \psi_{\lambda}^1(\mu, k) e^{\lambda^2+\mu^2} \Gamma(k + \frac{1}{2}). \end{aligned} \quad (26)$$

This concludes the deduction.  $\square$

## 6 Double H double prime

Let  $\mathcal{H}''$  be the double degeneration of the double affine Hecke algebra:

$$\mathcal{H}'' = \langle \partial, x, s \mid sxs = -x, \quad s\partial s = -\partial, \quad [\partial, x] = 1 + 2ks \rangle. \quad (27)$$

Its *polynomial representation*  $\rho : \mathcal{H}'' \rightarrow \text{End}(\mathcal{P})$  in  $\mathcal{P} = \mathbf{C}[x]$  is as follows:

$$\rho(x) = \text{multiplication by } x, \quad \rho(s) = s, \quad \rho(\partial) = \mathcal{D},$$

where  $s$  is the reflection  $f \mapsto f^t$ ,  $D$  is the Dunkl operator. The first two of the defining relations of  $\mathcal{H}''$  are satisfied thanks to Lemma 3.1. As to the last,

$$\begin{aligned} [\mathcal{D}, x] &= \left(\frac{\partial}{\partial x} - \frac{k}{x}(s-1)\right)x - x\left(\frac{\partial}{\partial x} - \frac{k}{x}(s-1)\right) \\ &= x\frac{\partial}{\partial x} + 1 - x\frac{k}{x}(-s-1) - x\frac{\partial}{\partial x} + x\frac{k}{x}(s-1) = 1 + 2ks. \end{aligned} \quad (28)$$

**Theorem 6.1** (a) Any nonzero finite linear combination  $H = \sum_{m,n,\epsilon} c_{m,n,\epsilon} x^m \partial^n s^\epsilon$ , where

$n, m \in \mathbf{Z}_+$ ,  $\epsilon \in \{0, 1\}$ , acts as a nonzero operator in  $\mathcal{P}$ .

(b) Any element  $H \in \mathcal{H}''$  can be uniquely expressed in the form  $H = \sum_{m,n,\epsilon} x^m \partial^n s^\epsilon$ . The representation  $\rho$  is faithful for any  $k$ .

*Proof.* Let  $H = \sum_{n,\epsilon} f_n \partial^n s^\epsilon = \sum_{m=0}^N (g_m \partial^m)(1+s) + \sum_{m=0}^M (h_m \partial^m)(1-s)$  for polynomials  $f_m, g_m,$

and  $h_m$ . We may assume that and at least one of the leading coefficients  $g_N(x)$ ,  $h_M(x)$  is nonzero. Then  $\rho(H) = L_+(1+s) + L_-(1-s)$  for differential operators  $L_+ = g_N(x)(\frac{\partial}{\partial x})^N + \dots$  and  $L_- = h_M(x)(\frac{\partial}{\partial x})^M + \dots$  modulo differential operators of lower orders. Applying  $\rho(H)$  to even and odd functions, we get that  $\rho(H) = 0$  implies that both  $L_+$  and  $L_-$  have infinite dimensional spaces of eigenfunctions. This is impossible. Claim (a) is verified.

Concerning (b), any element  $H \in \mathcal{H}''$  can be obviously expressed in the desired form. Such expression is unique and the representation  $\rho$  is faithful thanks to (a).  $\square$

The theorem is the key point of the representation theory of the double H. It is a variant of the so-called PBW theorem. There are not many algebras in mathematics and physics possessing this property. All have important applications. The double Hecke algebra is one of them.

Next, we study the irreducibility of  $\rho$ .

**Lemma 6.2** The Dunkl operator  $\mathcal{D}$  has only one eigenvalue in  $\mathcal{P}$ , namely,  $\lambda = 0$ . If  $k \neq -1/2 - n$  for any  $n \in \mathbf{Z}_+$ , then  $\mathcal{D}$  has a unique (up to a constant) eigenfunction in  $\mathcal{P}$ , the constant function 1. When  $k = -1/2 - n$  for  $n \in \mathbf{Z}_+$ , the space of 0-eigenfunctions is  $\mathbf{C} + \mathbf{C}x^{2n+1}$ .

*Proof.* Let  $p(x) \in \mathcal{P}$  be an eigenfunction for  $\mathcal{D}$ . Since  $\mathcal{D}$  lowers the degree of any polynomial by 1, we have  $\mathcal{D}^{m+1}p = 0$ , where  $m = \deg p$ . Therefore all eigenvalues of  $\mathcal{D}$  are zero. Representing  $p$  as the sum  $p(x) = p^0(x) + p^1(x)$  of even  $p^0$  and odd  $p^1$ ,  $\mathcal{D}p = \frac{\partial}{\partial x}p^0 + (\frac{\partial}{\partial x}p^1 + \frac{2k}{x}p^1) = 0$ . Both the even and the odd parts of this expression have to be zero. Hence  $\frac{\partial}{\partial x}p^0 = 0$  and  $\frac{\partial}{\partial x}p^1 + \frac{2k}{x}p^1 = 0$ . Therefore  $p^0 = \text{Const}$ . Setting  $p^1(x) = \sum a_l x^{2l+1}$ ,

$$(\frac{\partial}{\partial x} + \frac{2k}{x})p^1(x) = \sum a_l (2l + 1 + 2k)x^{2l} = 0.$$

If  $k \neq -1/2 - n$  for any  $n \in \mathbf{Z}_+$  then  $a_l = 0$  for any  $l$ , i.e  $p^1 = 0$ . Otherwise  $k = -1/2 - n$  for a certain  $n \in \mathbf{Z}_+$  and  $p^1(x)$  is proportional to  $x^{2n+1}$ .  $\square$

### Theorem 6.3

(a) The representation  $\rho$  is irreducible if and only if  $k \neq -1/2 - n$  for any  $n \in \mathbf{Z}_+$ .

(b) If  $k = -1/2 - n$  for  $n \in \mathbf{Z}_+$ , then there exists a unique non-trivial  $\mathcal{H}''$ -submodule of  $\mathcal{P}$ , namely,  $(x^{2n+1}) = x^{2n+1}\mathcal{P}$ .

*Proof.* Let  $\{0\} \neq V \subset \mathcal{P}$  be a  $\mathcal{H}''$  submodule of  $\mathcal{P}$ . Let  $0 \neq v \in V$  and  $m = \deg v$ . Set  $\mathcal{P}^{(m)} = \{p \in \mathcal{P} \mid \deg p \leq m\}$ . We have  $V^{(m)} = V \cap \mathcal{P}^{(m)} \neq \{0\}$ . Then  $V^{(m)}$  is  $\mathcal{D}$  invariant. More exactly,  $\mathcal{D}(V^{(m)}) \subset V^{(m-1)}$ . Thus it contains an eigenfunction  $v_0$  of  $\mathcal{D}$ . If  $k \neq -1/2 - n$  for  $n \in \mathbf{Z}_+$  then Lemma 6.2 implies that  $v_0 = 1$ . So  $1 \in V$  and  $V \ni \rho(x^m)(1) = x^m$  for any  $m \in \mathbf{N}$ . This means that  $V = \mathcal{P}$  and completes (a).

If  $k = -1/2 - n$  for  $n \in \mathbf{Z}_+$  then Lemma 6.2 states that  $v_0 = c_1 + c_2 x^{2n+1} \in V$  for constants  $c_1, c_2$ . If  $c_1 \neq 0$  then  $s(v_0) + v_0 = 2c_1 \in V$  (the latter is  $s$ -invariant and  $V = \mathcal{P}$  as above. If  $v_0 = x^{2n+1} \in V$  then this argument gives that  $(x^{2n+1}) = x^{2n+1}\mathcal{P} \subset V$ . Moreover  $V$  cannot contain the polynomials of degree less than  $2n+1$ . Otherwise we can find a 0-eigenvector of  $\mathcal{D}$  in the space of such polynomials, which is impossible. Hence  $V = (x^{2n+1})$ .

The latter is invariant with respect to  $x$  and  $s$ . Its  $\mathcal{D}$ -invariance readily follows from the formula  $\mathcal{D}(x^l) = (l + (1 - (-1)^l)k)x^{l-1}$  considered in the range  $l \geq 2n+1$ .  $\square$

We can reformulate the theorem as follows. The polynomial representation has a non-trivial (proper)  $\mathcal{H}''$ -quotient if and only if  $k = -1/2 - n$  for  $n \in \mathbf{Z}_+$ . In the latter case, such quotient is unique, namely,  $V_{2n+1} = \mathcal{P}/(x^{2n+1})$ . Its dimension is  $2n+1$ .

Note that the subspace  $V_{2n+1}^0$  of  $V_{2n+1}$  generated by even polynomials is invariant with respect to the action of  $h = x \frac{\partial}{\partial x} + k + 1/2$  and  $e = x^2$ ,  $f = -\mathcal{L}/4$ , satisfying the defining relations of  $sl_2(\mathbf{C})$  (see Section 2). We get an irreducible representation of  $sl_2(\mathbf{C})$  of dimension  $n+1$ .

## 7 Algebraization

Let us use  $\mathcal{H}''$  to formalize the previous considerations and to switch to the standard terminology of the representation theory.

**(a) Inner product.** We call a representation  $V$  of  $\mathcal{H}''$  *pseudo-unitary* if it possesses a non-degenerate  $\mathbf{C}$ -bilinear form  $(u, w)$  such that  $(Hu, w) = (u, H^*v)$  for  $H \in \mathcal{H}''$  for the anti-involution

$$\partial^* = -\partial, \quad s^* = s, \quad x^* = x. \quad (29)$$

By anti-involution, we mean a  $\mathbf{C}$ -automorphism satisfying  $(AB)^* = B^*A^*$ . We call such form  $\star$ -invariant. Formulas (29) are compatible with the defining relations (27) of  $\mathcal{H}''$  and therefore can be extended to the whole  $\mathcal{H}''$ . This is straightforward. For instance,

$$[x^*, \partial^*] = [x, -\partial] = [\partial, x] = 1 + 2ks = 1 + 2ks^* = [\partial, x]^*.$$

We add “pseudo” because the pairing, generally speaking, is not supposed to be positive and the functions can be complex-valued.

The pairing  $\langle f, g \rangle = \int_{\mathbf{R}} f(x)g(x)|x|^{2k}dx$  gives an example, provided the existence of the integral. Taking real-valued functions, we make this inner product positive (no “pseudo”). Assuming that the functions are  $C^\infty$ , we need to examine the convergence at  $x = 0$  and  $x = \infty$ . If  $\operatorname{Re}(k) > -1/2$  then it suffices to take regular  $f$  at  $x = 0$ . At infinity,  $f(x)|x|^k$  has to be of type  $L^1(\mathbf{R})$ . Polynomials times the Gaussian  $e^{-x^2}$  are fine.

**(b) Gaussians.** A homomorphism  $\gamma : V \rightarrow W$  for two  $\mathcal{H}''$ -modules  $V, W$  is called a *Gaussian* if  $\gamma H = \tau(H)\gamma$  for the following automorphism  $\tau$  of  $\mathcal{H}''$  :

$$\tau(\partial) = \partial - 2x, \quad \tau(x) = x, \quad \tau(s) = s. \quad (30)$$

These formulas can be extended to an automorphism of  $\mathcal{H}''$ . Indeed,  $\tau(s)\tau(\partial)\tau(s) = s(\partial + 2x)s = -\partial - 2x = -\tau(\partial)$ , the same holds for  $x$ , and

$$[\tau(\partial), \tau(x)] = [d - 2x, x] = 1 + 2s = \tau(1 + 2s).$$

Note that 2 can be replaced by any constant  $\alpha \in \mathbf{C}$  in this definition. We get a family of automorphisms  $\tau_\alpha(\partial) = \partial - 2\alpha x$  of  $\mathcal{H}''$ . They lead to the following generalization of the master formula:

$$\int_{\mathbf{R}} \psi_\lambda(x) \psi_\mu(x) e^{-\alpha x^2} |x|^{2k} dx = \psi_\lambda\left(\frac{\mu}{\alpha}\right) e^{\frac{\lambda^2 + \mu^2}{\alpha}} \alpha^{-k} \Gamma(k + \frac{1}{2}).$$

Here  $\alpha > 0$  to ensure the convergence. The substitution  $u = \sqrt{\alpha}x$  readily makes it equivalent to (24): use that  $\psi_\lambda(x)$  is a function of the  $x\lambda$ . One can also follow the proof of the master formula employing  $e^{-\alpha x^2} \mathcal{D} e^{\alpha x^2} = \mathcal{D} + 2\alpha x$ .

If representations  $V, W$  are algebras of functions on the same set then  $\gamma$  can be assumed to be a function, to be more exact, the operator of multiplication by a function. For instance, the multiplication by  $e^{x^2}$  sends the polynomial representation  $\mathcal{P}$  to  $\mathcal{P}e^{\pm x^2}$ . The latter is a  $\mathcal{H}''$ -module too. Adding all integral powers of  $e^{x^2}$  to  $\mathcal{P}$  we make this multiplication an inner automorphism of the resulting algebra. However it is somewhat artificial. Algebraically, the resulting representation  $\mathcal{P}[e^{mx^2}, m \in \mathbf{Z}]$  is “too” reducible. Analytically, we mix together  $e^{x^2}$  and  $e^{-x^2}$ , functions with absolutely different behaviour at infinity. There are interesting examples of inner automorphisms  $\tau$ , but they are finite-dimensional.

**(c) Hankel transform.** Following Lemma 5.1, the *operator Hankel transform* is the following automorphism of  $\mathcal{H}''$  :

$$\sigma(s) = s, \quad \sigma(\partial) = -2x, \quad \sigma(2x) = \partial. \quad (31)$$

These relations are obviously compatible with the defining relations of  $\mathcal{H}''$ . Any homomorphism  $\mathcal{F} : V \rightarrow W$  of  $\mathcal{H}''$ -modules inducing  $\sigma$  on  $\mathcal{H}''$  can be called a Hankel transform. The main example so far is  $\mathcal{F} : \mathcal{P}e^{-x^2} \rightarrow \mathcal{P}e^{+x^2}$ , where we identify  $x$  and  $\lambda$  in 23. Indeed,  $\mathcal{F} \circ \mathcal{D} = -2x \circ \mathcal{F}$ ,  $\mathcal{F} \circ 2x = \mathcal{D} \circ \mathcal{F}$ ,  $\mathcal{F} \circ s = s \circ \mathcal{F}$  upon this identification.

It is interesting to interpret the master formula from this viewpoint. It is nothing else but the following identities for  $\mathbf{F} = e^{x^2} e^{\partial^2/4} e^{x^2}$ :

$$\mathbf{F}s = s\mathbf{F}, \quad \mathbf{F}\partial = -2x\mathbf{F}, \quad \mathbf{F}(2x) = \partial\mathbf{F}. \quad (32)$$

This means that  $\mathbf{F}$  is Hankel transform whenever it is well-defined. Relations (32) can be deduced directly from the defining relations. In the first place, check that  $[\partial, x^2] =$

$2x, [\partial^2, x] = 2\partial$ . Then get that  $[\partial, e^{x^2}] = 2xe^{x^2}$ ,  $[e^{\partial^2/4}, x] = \partial e^{x^2}/2$  and use it as follows:

$$\begin{aligned} e^{x^2} e^{\partial^2/4} e^{x^2} 2x &= e^{x^2} e^{\partial^2/4} (2x) e^{x^2} = e^{x^2} (\partial + 2x) e^{\partial^2} e^{x^2} = \\ (\partial - 2x + 2x) e^{x^2} e^{\partial^2} e^{x^2} &= \partial e^{x^2} e^{\partial^2} e^{x^2}, \\ e^{-x^2} e^{-\partial^2/4} e^{-x^2} 2x &= e^{-x^2} e^{-\partial^2/4} (2x) e^{-x^2} = e^{-x^2} (-\partial + 2x) e^{-\partial^2} e^{-x^2} = \\ (-\partial - 2x + 2x) e^{x^2} e^{\partial^2} e^{x^2} &= -\partial e^{x^2} e^{\partial^2} e^{x^2}. \end{aligned}$$

The commutativity of  $\mathbf{F}$  with  $s$  is obvious.

Note the following *braid identity* which can be proved similarly:

$$\mathbf{F} = e^{x^2} e^{\partial^2/4} e^{x^2} = e^{\partial^2} e^{x^2/4} e^{\partial^2}.$$

Actually we do need calculations from scratch because it suffices to use the nonsymmetric master formula Lemma 5.1 and the fact that  $\mathcal{P}$  is a faithful representation. For instance,  $\mathbf{F}(2x)\mathbf{F}^{-1}$  and  $\partial$  coincide in  $\mathcal{P}$ . The previous consideration shows that the former is an element of  $\mathcal{H}''$ . Hence they must coincide identically, i.e. in  $\mathcal{H}''$ .

A good demonstration of the convenience of such an algebraization will be the case of negative half-integral  $k$ . Before switching to this case, let us conclude the “analytic” theory calculating the inverse Hankel transform.

## 8 Inverse transform and Plancherel formula

Let  $\operatorname{Re} k > -1/2$ . We use  $\psi_\lambda(x)$  from (16).

$$(\mathcal{F}_{re}f)(\lambda) = \frac{1}{\Gamma(k+1/2)} \int_{-\infty}^{+\infty} \psi_\lambda(x) f(x) |x|^{2k} dx, \quad (33)$$

$$(\mathcal{F}_{im}g)(x) = \frac{1}{\Gamma(k+1/2)} \int_{-\infty}^{+\infty} \psi_x(-\lambda) g(\lambda) |\lambda|^{2k} d\lambda. \quad (34)$$

The first is nothing else but  $\mathcal{F}$  from (23). We just show explicitly that the integration is real.

Here we may consider  $\mathbf{C}$ -valued functions  $f$  on  $\mathbf{R}$  and  $g$  on  $\mathbf{C}$  respectively such that

$$f(x) = o(e^{cx}) \text{ at } x = \infty \forall c \in \mathbf{R} \text{ and } f \in g(\lambda) = o(e^{ci\lambda}) \text{ at } \lambda = i\infty \forall c \in \mathbf{R}.$$

Restricting ourselves with the polynomials times the Gaussian,

$$\mathcal{F}_{re} : \mathbf{C}[x]e^{-x^2} \rightarrow \mathbf{C}[\lambda]e^{\lambda^2}, \quad \mathcal{F}_{im} : \mathbf{C}[\lambda]e^{\lambda^2} \rightarrow \mathbf{C}[x]e^{-x^2}.$$

The first map is an isomorphism. Let us discuss the latter.

Let  $p(x) \in \mathbf{C}[x]$ . Applying the master formula to  $p_i(x) = p(ix)$ ,

$$\frac{1}{\Gamma(k+1/2)} \int_{-\infty}^{+\infty} \psi_\lambda(x) p(ix) e^{-x^2} |x|^{2k} dx = \mathcal{F}_{re}(e^{-x^2} p(ix)) = e^{\lambda^2} \exp((\mathcal{D}^\lambda)^2/4)(p(i\lambda)).$$

Since  $\mathcal{D} \circ I = iI \circ \mathcal{D}$  for  $I(f)(x) = f(ix)$ ,

$$\frac{(\mathcal{D}^\lambda)^2}{4} p(i\lambda) = \left( \frac{-(\mathcal{D}^u)^2}{4} p(u) \right) \Big|_{u=i\lambda}.$$

Now we replace  $\lambda \mapsto i\lambda$ , use that  $\psi_{i\lambda}(x) = \psi_\lambda(ix)$ , and then integrate by substitution using  $z = ix$ . The resulting formula reads:

$$\frac{1}{\Gamma(k+1/2)} \int_{-\infty}^{+\infty} \psi_\lambda(ix) p(ix) e^{-x^2} |x|^{2k} dx = e^{-\lambda^2} \exp(-(\mathcal{D}^\lambda)^2/4) (p(-\lambda)) \quad (35)$$

$$= \frac{1}{i \Gamma(k+1/2)} \int_{-i\infty}^{+i\infty} \psi_\lambda(z) p(z) e^{z^2} |z|^{2k} dz. \quad (36)$$

Switching to  $\lambda$ ,  $\mathcal{F}_{im}(e^{\lambda^2} p(\lambda)) = e^{-x^2} \exp(-(\mathcal{D}^x)^2/4) (p(x))$ .

We come to the inversion theorem.

**Theorem 8.1 (Inversion Formula)**  $\mathcal{F}_{im} \circ \mathcal{F}_{re} = \text{id}$  in  $\mathbf{C}[x]e^{-x^2}$ ,  $\mathcal{F}_{re} \circ \mathcal{F}_{im} = \text{id}$  in  $\mathbf{C}[\lambda]e^{\lambda^2}$ .

*Proof:*  $\mathcal{F}_{im} \circ \mathcal{F}_{re}(e^{-x^2} p(x)) = e^{-x^2} \exp(-\mathcal{D}^2/4) \exp(\mathcal{D}^2/4) (p(x)) = e^{-x^2} p(x)$ . The second formula is analogous.  $\square$

There is a simple “algebraic” proof based on the facts that the transform  $\mathcal{F}_{im} \circ \mathcal{F}_{re}$  sends  $\mathcal{D} \mapsto \mathcal{D}$ ,  $2x \mapsto 2x$ ,  $s \mapsto s$ . Thanks to the irreducibility of  $\mathbf{C}[x]e^{\pm x^2}$ , we may apply the Schur lemma. However the spaces are infinite dimensional, so a minor additional consideration is necessary. We will skip it because it practically coincides with that from the proof of the Plancherel formula.

Provided  $\text{Re } k > -\frac{1}{2}$ , the inner products

$$\langle f, g \rangle_{re} = \int_{-\infty}^{+\infty} f(x) g(x) |x|^{2k} dx, \quad \langle f, g \rangle_{im} = \frac{1}{i} \int_{-i\infty}^{+i\infty} f(\lambda) g(-\lambda) |\lambda|^{2k} dx \quad (37)$$

are non-degenerate respectively in  $\mathbf{C}[x]e^{-x^2}$  and  $\mathbf{C}[\lambda]e^{\lambda^2}$ .

It is obvious when  $\mathbf{R}[x], \mathbf{R}[\lambda]$  are considered instead of  $\mathbf{C}[x], \mathbf{C}[\lambda]$  and  $k \in \mathbf{R}$ . Indeed, both forms become positive in this case. Concerning the second, use that  $g(-\lambda) = \overline{g(\lambda)}$  for a real polynomial  $g$ .

For complex-valued functions and  $k \in \mathbf{C}$ , the claim requires proving. Let us use the irreducibility of  $\mathcal{H}''$ -modules  $\mathbf{C}[x]e^{-x^2}$  and  $\mathbf{C}[x]e^{x^2}$ , which is equivalent to the irreducibility of the polynomial representation  $\mathcal{P} = \mathbf{C}[x]$ , which we already know for  $\text{Re } k > -\frac{1}{2}$ . Then the radical of the form  $\langle \cdot, \cdot \rangle_{re}$  is a submodule of  $\mathbf{C}[x]e^{-x^2}$ . It does not coincide with the whole space, since

$$\langle e^{-x^2}, e^{-x^2} \rangle_{re} = \int_{\mathbf{R}} e^{-2x^2} |x|^{2k} dx = (\sqrt{2})^{-2k-1} \Gamma(k + \frac{1}{2}). \quad (38)$$

The same argument works for the imaginary integration.



**Theorem 8.2 (Plancherel Formula)**

$$\langle f, g \rangle_{re} = \langle \widehat{f}, \widehat{g} \rangle_{im} \text{ for all } f, g \in \mathbf{C}[x]e^{-x^2}, \quad \widehat{f} = \mathcal{F}_{re}(f), \quad \widehat{g} = \mathcal{F}_{re}(g). \quad (39)$$

*Proof.* Setting  $\mathcal{P}_- = \mathbf{C}[x]e^{-x^2}$ , we need to check that  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{re}$  coincides with  $\langle f, g \rangle_1 = \langle \widehat{f}, \widehat{g} \rangle_{im}$  for all  $f, g \in \mathcal{P}_-$ . In the first place,  $\langle Hf, g \rangle_1 = \langle f, H^*g \rangle_1$  for any  $H \in \mathcal{H}''$ , i.e. this bilinear form is  $\star$ -invariant. Indeed,

$$\langle \mathcal{D}f, g \rangle_1 = \langle \widehat{\mathcal{D}f}, \widehat{g} \rangle_{im} = \langle -2x\widehat{f}, \widehat{g} \rangle_{im} = -\langle \widehat{f}, -2(-x)\widehat{g} \rangle_{im} = -\langle \widehat{f}, \widehat{\mathcal{D}g} \rangle_{im} = -\langle f, \mathcal{D}g \rangle_1.$$

Similarly,  $\langle 2xf, g \rangle_1 = \langle f, 2xg \rangle_1$  and  $\langle s(f), g \rangle_1 = \langle f, s(g) \rangle_1$ .

Setting  $\langle \cdot, \cdot \rangle_2 = \langle \cdot, \cdot \rangle - \langle \cdot, \cdot \rangle_1$ , we get  $\langle e^{-x^2}, e^{-x^2} \rangle_2 = 0$ . Cf. 38. It is a  $\star$ -invariant form as well. Let us demonstrate that it vanishes identically.

First,  $\langle e^{-x^2}, (\mathcal{D} - 2x)f \rangle = \langle -(\mathcal{D} + 2x)e^{-x^2}, f \rangle = \langle 0, f \rangle = 0$  for any  $f \in \mathcal{P}_-$  due to the  $\star$ -invariance. So it is applicable to  $\langle \cdot, \cdot \rangle_2$  too. Second,  $\mathbf{C}e^{-x^2} \cap (\mathcal{D} - 2x)\mathcal{P}_- = \emptyset$  because  $\langle e^{-x^2}, e^{-x^2} \rangle \neq 0$ . Third,  $\mathcal{P}_- = \mathbf{C}e^{-x^2} \oplus (\mathcal{D} - 2x)\mathcal{P}_-$ . Really, the dimension of  $(\mathcal{D} - 2x)\mathcal{P}_{n-}$  for  $\mathcal{P}_n = \mathbf{C}[1, x, \dots, x^n]e^{-x^2}$  is  $n + 1$  since the kernel of the operator  $\mathcal{D} - 2x$  in  $\mathcal{P}_-$  is zero. However  $(\mathcal{D} - 2x)\mathcal{P}_{n-} \subset \mathcal{P}_{(n+1)-}$ . Therefore  $(\mathcal{D} - 2x)\mathcal{P}_{n-} = \mathcal{P}_{(n+1)-}$ . Finally,

$$\langle e^{-x^2}, \mathcal{P}_- \rangle_2 = \langle e^{-x^2}, e^{-x^2} + (\mathcal{D} - 2x)\mathcal{P}_- \rangle_2 = 0$$

and  $e^{-x^2}$  belongs to the radical of  $\langle \cdot, \cdot \rangle_2$ . Since the module  $\mathcal{P}_-$  is irreducible, the radical has to coincide with the whole  $\mathcal{P}_-$ .  $\square$

Taking real  $k$ , the forms  $\langle \cdot, \cdot \rangle_{re}$   $\langle \cdot, \cdot \rangle_{im}$  are positive on  $\mathbf{R}[x]e^{-x^2}$  and  $\mathbf{R}[\lambda]e^{\lambda^2}$ . The Plancherel formula allows us to complete the function spaces extending the Fourier transforms  $\mathcal{F}_{re}, \mathcal{F}_{im}$  to the spaces of square integrable real-valued functions with respect to the “Bessel measure”:

$$\mathcal{L}^2(\mathbf{R}, |x|^{2k}dx) \rightarrow \mathcal{L}^2(i\mathbf{R}, |\lambda|^{2k}d\lambda) \rightarrow \mathcal{L}^2(\mathbf{R}, |x|^{2k}dx).$$

The inversion and Plancherel formulas remain valid.

Here we assume that  $k > -\frac{1}{2}$ . Let us discuss the case of negative half-integers.

## 9 Finite-dimensional case

Let  $k = -n - \frac{1}{2}$  for  $n \in \mathbf{Z}_+$ . Then  $V_{2n+1} = \mathcal{P}/(x^{2n+1})$  is an irreducible representation of  $\mathcal{H}''$ . The elements of  $V_{2n+1}$  can be identified with polynomials of degree less than  $2n + 1$ .

**Theorem 9.1** *Finite-dimensional representations of  $\mathcal{H}''$  exist only as  $k = -n - 1/2$  for  $n \in \mathbf{Z}_+$ . Given such  $k$ , the algebra  $\mathcal{H}''$  has a unique finite-dimensional irreducible representation up to isomorphisms, namely,  $V_{2n+1}$ .*

*Proof.* We will use that

$$[h, x] = x, [h, \partial] = -\partial \text{ for } h = (x\partial + \partial x)/2. \quad (40)$$

It readily follows from the defining relations of  $\mathcal{H}''$ . Actually (40) determines a super Lie algebra. One may use a general theory of such Lie algebras. However in this particular case a reduction to  $sl_2$  is more than sufficient.

Note that  $h$  is  $x\frac{\partial}{\partial x} + k + 1/2$  in the polynomial representation. Since the latter is faithful, (40) is exactly the claim that  $x, \mathcal{D}$  are homogeneous operators of degree  $\pm 1$ , which is obvious.

We will employ that  $e = x^2$ ,  $f = -\partial^2/4$ , and  $h$  satisfy the defining relations of  $sl_2(\mathbf{C})$ . Namely,  $[e, f] = h$  because

$$[\partial^2, x^2] = [\partial^2, x]x + x[\partial^2, x] = 2\partial x + x(2\partial),$$

and the relations  $[h, e] = 2e$ ,  $[h, f] = -2f$  readily result from (40). Cf. Section 6.

Let  $V$  be a finite-dimensional representation of  $\mathcal{H}''$ . Then the subspaces  $V^0, V^1$  of  $V$  formed respectively by  $s$ -invariant and  $s$ -anti-invariant vectors are preserved by  $h$ ,  $e$ , and  $f$ . So they are  $sl_2(\mathbf{C})$ -modules. One gets

$$\partial x = h + k + 1/2, \quad x\partial = h - k - 1/2 \text{ in } V^0,$$

and the other way round in  $V^1$ .

Let us check that  $k \in -1/2 - \mathbf{Z}_+$ . All  $h$ -eigenvalues in  $V$  are integers thanks to the general theory of finite-dimensional representations of  $sl_2(\mathbf{C})$ . We pick a nonzero  $h$ -eigenvector  $v \in V$  with the maximal possible eigenvalue  $m$ . Then  $m \in \mathbf{Z}_+$  (the theory of  $sl_2$ ) and  $x(v) = 0$  because the latter is an  $h$ -eigenvector with the eigenvalue  $m + 1$ . Hence  $\partial x(v) = 0$ ,  $m + k + 1/2 = 0$ , and  $k = -1/2 - m$ .

Let  $U^0$  be a nonzero irreducible  $sl_2(\mathbf{C})$ -submodule of  $V^0$ . The spectrum of  $h$  in  $U^0$  is  $\{-n, -n + 2, \dots, n - 2, n\}$  for an integer  $n \geq 0$ . Let  $v_l \neq 0$  be an  $h$ -eigenvector with the eigenvalue  $l$ . If  $e(v) = 0$  then  $v = cv_n$  for a constant  $c$ , and if  $f(v) = 0$  then  $v = cv_{-n}$ .

Let us check that  $\partial x(v_n) = 0$ ,  $x\partial(v_{-n}) = 0$ , and

$$\partial x(v_l) \neq 0 \text{ for } l \neq n, \quad x\partial(v_l) \neq 0 \text{ for } l \neq -n.$$

Both operators,  $\partial x$  and  $x\partial$ , obviously preserve  $U^0$  :

$$\partial x(v_l) = (l + k + 1/2)v_l, \quad x\partial(v_l) = (l - k - 1/2)v_l.$$

Hence,

$$\partial^2 x^2(v_l) = ((\partial x)^2 + (1 - 2k)(\partial x))(v_l) = (l + k + 1/2)(l - k + 3/2)v_l.$$

Setting  $l = n$ , we get that  $(n + k + 1/2)(n - k + 3/2) = 0$  and  $k = -1/2 - n$ , because  $k < 0$  and  $n - k + 3/2 > 0$ . Thus  $\partial x(v_n) = 0$ . The case of  $x\partial$  is analogous.

The next claim is that  $x(v_n) = 0$ ,  $\partial(v_{-n}) = 0$ . Indeed,  $x(v') = 0$  and  $\partial(v') = 0$  for  $v' = x(v_n)$ . Therefore

$$0 = [\partial, x](v') = (1 + 2ks)(v') = (1 - 1 + n)v' = nv'.$$

This means that either  $v' = 0$  or  $n = 0$ . In the latter case,  $v'$  is proportional to  $v_0$  and therefore  $v' = x(v_0) = 0$  as well. Similarly,  $\partial(v_{-n}) = 0$ .

Now we use the formula

$$\partial(x^2(v_l)) = x(2 + x\partial)(v_l) = (2 + l - k - 1/2)x(v_l) = (2 + l + n)x(v_l),$$

and get that  $x(v_l) \in \partial(U^0)$  for any  $-n \leq l \leq n$ . Hence  $U = U^0 + \partial(U^0)$  is  $x$ -invariant. It is obviously  $\partial$ -invariant and  $s$ -invariant ( $\Leftarrow \partial(V^0) \subset V^1$ ). Also the sum is direct.

Finally,  $U$  is a  $\mathcal{H}''$ -module and has to coincide with  $V$  because the latter was assumed to be irreducible. The above formulas are sufficient to establish a  $\mathcal{H}''$ -isomorphism  $U \simeq V_{2n+1}$ . Explicitly, the  $h$ -eigenvectors  $x^i(v_{-n}) \in U$  will be identified with the monomials  $x^i \in V_{2n+1}$ .  $\square$

Let us discuss the Hankel transform and related structures in the case of  $V_{2n+1}$ . We follow Section 7.

**(a) Form.** To make  $\star$  “inner” we have to construct a non-degenerate bilinear pairing  $(u, v)$  on  $V_{2n+1}$  such that  $(Hu, v) = (u, H^*v)$ . Here it is:

$$\forall f, g \in V_{2n+1} \text{ set } (f, g) = \text{Res}(f(x)g(x)x^{-2n-1}), \text{ where } \text{Res}(\sum a_i x^i) = a_{-1}. \quad (41)$$

The pairing is non-degenerate, because if  $f = ax^l + \text{lower order terms}$ , where  $a \neq 0$  and  $0 \leq l \leq 2n$ , then  $(f, x^{2n-l}) = a$ .

We introduce a scalar product  $(f, g)_0 = \text{Res}(fg)$  for polynomials in terms of  $x$  and  $x^{-1}$ . Denoting the conjugate of an operator  $A$  with respect to this pairing by  $A^\circ$ ,

$$s^\circ = -s, \quad x^\circ = x, \quad \frac{\partial}{\partial x}^\circ = -\frac{\partial}{\partial x}, \quad \text{and} \quad x^{2n+1}\mathcal{D}^\circ x^{-2n-1} = -\mathcal{D}.$$

The relation  $x^\circ = x$  is obvious. Concerning  $\frac{\partial}{\partial x}^\circ = -\frac{\partial}{\partial x}$ , it follows from the property  $\text{Res}(df/dx) = 0$  for any polynomial  $f(x, x^{-1})$ . The formula  $s^\circ = -s$  results from  $\text{Res}(s(f(x))) = -\text{Res}(f(x))$ .

Switching from  $^\circ$  to  $\star$ , we have

$$(sf, g) = \text{Res}(s(f)gx^{-2n-1}) = -\text{Res}(fs(g)s(x^{-2n-1})) = -(-1)^{2n+1}(fs(g)x^{-2n-1}) = (f, s(g)).$$

Finally,

$$\begin{aligned} \mathcal{D}^\circ &= \left(\frac{\partial}{\partial x} + \frac{k}{x}(1-s)\right)^\circ = -\frac{\partial}{\partial x} + (1+s)\frac{k}{x}, \\ x^{-2k}\mathcal{D}^\circ x^{2k} &= -\frac{\partial}{\partial x} - \frac{2k}{x} + \frac{k}{x}(1+s) = -\frac{\partial}{\partial x} + \frac{k}{x}(-1+s) = -\mathcal{D}. \end{aligned} \quad (42)$$

The first equality on the second line holds because  $\frac{k}{x}x^{2k} = kx^{-2n-2}$  is an even function, and thus it commutes with the action of  $s$ . Finally

$$(\mathcal{D}f, g) = (\mathcal{D}f, x^{2k}g)_0 = (f, x^{2k}x^{-2k}\mathcal{D}^\circ x^{2k}(g))_0 = -(f, x^{2k}\mathcal{D}g)_0 = -(f, \mathcal{D}g). \quad \square$$

(b) **Gaussian.** The Gaussian does not exist in polynomials but of course can be introduced as a power series  $e^{x^2} = \sum_{m=0}^{\infty} (x^2)^m/m!$  in the algebra of formal series  $\mathbf{C}[[x]]$ , a completion of the polynomial representation. The conjugation by this series induces  $\tau$  on  $\mathcal{H}''$ . Its inverse is  $e^{-x^2} = \sum_{m=0}^{\infty} (-x^2)^m/m!$ . The multiplication by the Gaussian does not preserve the space of polynomials but is well-defined on  $V_{2n+1}$  because  $\forall f \in V_{2n+1}$  we have  $x^m f = 0$  for  $m \geq 2n+1$ . Finally,

$$\gamma^{\pm} = \sum_{m=0}^{2n} (\pm x^2)^m/m!.$$

(c) **Hankel transform.** The operator  $\mathcal{D}$  is nilpotent in  $V_{2n+1}$  because it lowers the degree of  $f \in V_{2n+1}$  by one. Therefore the operators  $\exp(\pm \mathcal{D}^2/4) \in \mathbf{C}[[\mathcal{D}]]$  are well-defined in this representation as well as the Gaussians. It suffices to take  $\sum_{m=0}^{2n} (\pm (D/2)^2)^m/m!$ . Thus we may set

$$\mathbf{F} = e^{x^2} e^{\frac{\mathcal{D}^2}{4}} e^{x^2} \quad \text{in } V_{2n+1}. \quad (43)$$

**Proposition 9.2** *The map  $\mathcal{F}$  is a Hankel transform on  $V_{2n+1}$ , i.e.  $\mathbf{F} \circ \mathcal{D} = -\mathbf{F} \circ 2x$ ,  $\mathbf{F} \circ 2x = \mathcal{D} \circ \mathbf{F}$ ,  $\mathbf{F}s = \mathbf{F}s$ . These relations fix it uniquely up to proportionality.*

*Proof.* We already know that  $\mathbf{F}$  is a Hankel transform (the previous section). If  $\tilde{\mathbf{F}}$  is another one then the ratio  $\tilde{\mathbf{F}}\mathbf{F}^{-1}$  commutes with  $x, \mathcal{D}$ , and  $s$  because of the very definition. Since  $V_{2n+1}$  is irreducible (and finite dimensional) we get that  $\tilde{\mathbf{F}}$  is proportional to  $\mathbf{F}$ .  $\square$

## 10 Truncated Bessel functions

Recall that  $\mathcal{D}$  has only one eigenvalue in  $V_{2n+1}$ , namely, 0. Therefore we cannot define the  $\psi_{\lambda}$  as an eigenfunction of  $\mathcal{D}$  in  $V_{2n+1}$  any longer. Instead, it will be introduced as the kernel of the Hankel transform.

Any linear operator  $A : V_{2n+1}^x \rightarrow V_{2n+1}^{\lambda}$  (the upper index indicates the variable) is a matrix. It means that

$$\begin{aligned} A(f)(\lambda) &= (f(x), \alpha(x, \lambda)) = \text{Res}(f(x)\alpha(x, \lambda)x^{-2n-1}), \text{ where} \\ \alpha(x, \lambda) &= \sum_{l,m=0}^{2n} c_{l,m} x^l \lambda^m = \sum_{l=0}^{2n} x^{2n-l} A(x^l). \end{aligned} \quad (44)$$

So here the kernel  $\alpha(x, \lambda)$  is uniquely defined by  $A$  and vice versa.

The *truncated  $\psi$ -function* is the kernel of  $\mathbf{F}$  :

$$\mathbf{F}(f)(\lambda) = (f(x), \psi_{\lambda}(x)) = \text{Res}(f(x)\psi_{\lambda}(x)x^{-2n-1}). \quad (45)$$

There is a somewhat different approach. Let us use that the relations from Lemma 9.2 determine  $\mathbf{F}$  uniquely up to proportionality. These relations are equivalent to the following properties of  $\psi_\lambda(x)$  :

$$\mathcal{D}\psi_\lambda(x) = 2\lambda\psi_\lambda(x) \mod (x^{2n+1}, \lambda^{2n+1}), \quad \psi_\lambda(x) = \psi_x(\lambda), \quad \psi_\lambda(s(x)) = \psi_{s(\lambda)}(x). \quad (46)$$

Let us solve the first equation. Setting  $\psi_\lambda(x) = \sum_{l,m=0}^{2n} c_{l,m} x^l \lambda^m$ ,

$$\begin{aligned} & \sum_{l=1, m=0}^{l=2n, m=2n} c_{l,m} (l + (1 - (-1)^l)(-n - \frac{1}{2})) x^{l-1} \lambda^m \\ &= \sum_{l=0, m=0}^{l=2n, m=2n-1} 2c_{l,m} x^l \lambda^{m+1} \mod (x^{2n+1}, \lambda^{2n+1}), \\ & c_{l,m} = \frac{2}{l + (1 - (-1)^l)(-n - 1/2)} c_{l-1, m-1} \text{ for } 2n \geq l > 0, \quad 2n \geq m > 0, \\ & \text{where } c_{l,0} = 0 = c_{2n,m} \text{ for } l > 0, \quad m < 2n. \end{aligned} \quad (47)$$

Using the  $x \leftrightarrow \lambda$  symmetry, we conclude that  $c_{l,0} = 0 = c_{0,l}$  for nonzero  $l$  and  $c_{l,m} = 0$  for  $l \neq m$ . Thus

$$\psi_\lambda(x) = g_n(\lambda x) \text{ for } g_n = \sum_{l=0}^{2n} c_l t^l, \quad c_l = c_{l,l},$$

where the coefficients are given by (47).

Finally,  $g_n(t) = f_n(t) + (1/2)df_n/dt$  for the *truncated Bessel function*  $f_n(t) = \sum_{m=0}^n c_{2m} t^{2m}$  which is an even solution of the truncated Bessel equation (cf. Section 1):

$$\frac{d^2 f}{dt^2}(t) + 2k \frac{1}{t} \frac{df}{dt}(t) - 4f(t) = 0 \mod (t^{2n}), \quad k = -n - 1/2. \quad (48)$$

This equation is sufficient to determine the coefficients of  $f_n$  uniquely for any constant term  $c_0 = c_{0,0}$ . They are given by the same formula (3) till  $c_{2n}$  up to proportionality. This can be checked directly using explicit formulas which will be discussed next.

Still  $c_0$  remains arbitrary. Recall that  $\psi$  was initially introduced as the kernel of  $\mathbf{F}$ . So it comes with its own normalization. Let us calculate its  $c_0$ . One gets:

$$\begin{aligned} \mathbf{F}(e^{-x^2}) &= e^{\lambda^2} \exp(\mathcal{D}^2/4)(e^{\lambda^2} e^{-\lambda^2}) = e^{\lambda^2} \exp(\mathcal{D}^2/4)(1) = e^{\lambda^2}, \text{ so} \\ \mathbf{F}(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \dots + (-1)^n \frac{x^{2n}}{n!}) &= 1 + \frac{\lambda^2}{1!} + \dots + \frac{\lambda^{2n}}{n!}. \end{aligned} \quad (49)$$

Here the transform of 1 is proportional to  $\lambda^{2n}$  since the latter has to be an eigenfunction of  $\lambda$ , i.e. the solution of the equation  $\lambda \mathbf{F}(1) = 0$  in  $V_{2n+1}^\lambda$ . Similarly,  $\mathbf{F}(x^l) = (\mathcal{D}^\lambda/2)^l \mathbf{F}(1)$  is proportional to  $\lambda^{2n-l}$  for  $0 \leq l \leq 2n$ . Thus (49) leads to the relations

$$\mathbf{F}(x^{2m}) = (-1)^m \frac{m!}{(n-m)!} \lambda^{2n-2m}. \quad (50)$$

For instance,  $\mathbf{F}(x^{2n}) = (-1)^n n!$ . This is exactly the coefficient  $c_0$  above.

We obtain that the normalization serving the truncated Hankel transform is

$$\psi_\lambda(0) = -n!, \quad c_0 = g_n(0) = f_n(0) = -n!. \quad (51)$$

Formula (50) also results in

$$\mathbf{F}(x^{2m+1}) = \mathbf{F}(x(x^{2m})) = (\mathcal{D}/2)\mathbf{F}(x^{2m}) = (-1)^m \frac{m!}{(n-m-1)!} \lambda^{2n-2m-1}. \quad (52)$$

Substituting,

$$\begin{aligned} \psi_\lambda(x) &= \sum_{m=0}^n x^{2n-2m} \mathbf{F}(x^{2m}) + \sum_{m=0}^{n-1} x^{2n-2m-1} \mathbf{F}(x^{2m+1}) = f_n(x\lambda) + \frac{1}{2} f'_n(x\lambda) \text{ for} \\ f_n(t) &= \sum_{m=0}^n \frac{(-1)^m m!}{(n-m)!} t^{2n-2m} = \sum_{m=0}^n \frac{(-1)^{n-m} (n-m)!}{m!} t^{2m}. \end{aligned} \quad (53)$$

It is exactly the solution of (48) with the *truncated normalization*  $f_n(0) = (-1)^n n!$ .

**Truncated inversion.** Concluding the consideration of the case  $k = -n - \frac{1}{2}$  for  $n \in \mathbf{Z}_+$ , let us discuss the inversion. We have the following transformations and scalar products:

$$\begin{aligned} \mathbf{F}_+ : \mathbf{C}[x]/(x^{2n+1}) &\rightarrow \mathbf{C}[\lambda]/(\lambda^{2n+1}), & \mathbf{F}_+(f) &= \text{Res}(f(x)\psi_\lambda(x)x^{-2n-1}), \\ \mathbf{F}_- : \mathbf{C}[\lambda]/(\lambda^{2n+1}) &\rightarrow \mathbf{C}[x]/(x^{2n+1}), & \mathbf{F}_-(f) &= \text{Res}(f(\lambda)\psi_x(-\lambda)\lambda^{-2n-1}). \\ \langle f, g \rangle_+ &= \text{Res}(f(x)g(x)x^{-2n-1}), & f, g &\in \mathbf{C}[x]/(x^{2n+1}); \\ \langle f, g \rangle_- &= \text{Res}(f(-\lambda)g(\lambda)\lambda^{-2n-1}), & f, g &\in \mathbf{C}[\lambda]/(\lambda^{2n+1}). \end{aligned} \quad (54)$$

Here  $\mathbf{F}_+(f) = \mathbf{F}(f) = \hat{f}$  in the notation above. The transform  $\mathbf{F}_-(f)$  coincides with  $\mathbf{F}_+^\lambda(f)$  for even  $f(\lambda)$  and with  $-\mathbf{F}_+^\lambda(f)$  for odd  $f(\lambda)$ .

We can follow the “analytic” case and check that  $\mathbf{F}_- \circ \mathbf{F}_+$  commutes with  $\mathcal{D}, x, s$ . Hence it is the multiplication by a constant thanks to the irreducibility of  $V_{2n+1}$ . The constant is  $\mathbf{F}_- \circ \mathbf{F}_+(1)$  and can be readily calculated. It is equally simple to calculate all  $\mathbf{F}_- \circ \mathbf{F}_+(x^l)$  using (50) and (52). For instance,

$$\begin{aligned} \mathbf{F}_- \circ \mathbf{F}_+(x^{2m}) &= \mathbf{F}_- \left( \frac{(-1)^m m!}{(n-m)!} \lambda^{2n-2m} \right) = \\ &= \frac{(-1)^m m!}{(n-m)!} \frac{(-1)^{n-m} (n-m)!}{m!} x^{2m} = (-1)^n x^{2m}. \end{aligned}$$

Thus the truncated inversion reads:

$$\mathbf{F}_- \circ \mathbf{F}_+ = (-1)^n \text{id} = \mathbf{F}_+ \circ \mathbf{F}_-.$$

Concerning the Plancherel formula, we may use the proportionality of the forms  $\langle f, g \rangle_+$  and  $\langle \widehat{f}, \widehat{g} \rangle_-$  for  $f, g \in V_{2n+1}$  and their transforms  $\widehat{f} = \mathbf{F}(f), \widehat{g} = \mathbf{F}(g)$ . It results from the irreducibility of  $V_{2n+1}$ . A direct calculation is simple as well. Let

$$\begin{aligned}\langle f, f \rangle_+ &= \langle f, f \rangle = \sum_{l=0}^{2n} a_l a_{2n-l} \text{ for } f = \sum_{l=0}^{2n} a_l x^l, \\ \langle g, g \rangle_- &= \sum_{l=0}^{2n} (-1)^l b_l b_{2n-l} \text{ for } g = \mathbf{F}(f) = \sum_{l=0}^{2n} b_l \lambda^l.\end{aligned}\tag{55}$$

It is easy to check that

$$b_l b_{2n-l} = (-1)^{l+n} a_l a_{2n-l}.$$

Indeed, using (50) and (52):

$$\begin{aligned}b_{2m} b_{2n-2m} &= (-1)^m a_{2m} \frac{m!}{(n-m)!} (-1)^{n-m} a_{2n-2m} \frac{(n-m)!}{m!} \\ &= (-1)^n a_{2m} a_{2n-2m}, \\ b_{2m+1} b_{2n-2m-1} &= (-1)^m a_{2m+1} \frac{m!}{(n-m-1)!} (-1)^{n-m-1} a_{2n-2m-1} \frac{(n-m-1)!}{m!} \\ &= (-1)^{n-1} a_{2m+1} a_{2n-2m-1}.\end{aligned}$$

We get the truncated Plancherel formula:

$$\langle \widehat{f}, \widehat{g} \rangle_- = (-1)^n \langle f, g \rangle_+.$$

The above consideration proves the coincidence for  $f = g$ , i.e. for the corresponding quadratic forms. It is of course sufficient.

## References

- [C1] Cherednik, I.: Difference Macdonald-Mehta conjectures. IMRN **10**, 449–467 (1997).
- [C2] Cherednik, I.: One-dimensional double Hecke algebras and Gaussians, CIME (2000).
- [D] Dunkl, C.F.: Differential-difference operators associated to reflection groups, Trans. AMS. **311**, 167–183 (1989).
- [J] Jeu, M.F.E. de: The Dunkl transform, Invent. Math. **113**, 147–162 (1993).
- [L] Luke, J.: Integrals of Bessel functions, McGraw-Hill Book Company, New York-Toronto-London (1962).
- [O] Opdam, E.M.: Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group, Comp. Math. **85**, 333–373 (1993).